

GEOMETRY OF GENERIC MOISHEZON TWISTOR SPACES ON $4\mathbb{CP}^2$

II: DEGENERATE CASES

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ABSTRACT. We continue to study twistor spaces on the connected sum of four complex projective planes, whose anticanonical map is of degree two over the image. In particular, we determine the defining equation of the branch divisor of the anticanonical map in an explicit form. Together with previous two articles ([3] and [4]), this completes explicit description of all such twistor spaces.

1. INTRODUCTION

This paper is a sequel to an article [4], where we intensively studied twistor spaces on $4\mathbb{CP}^2$ which could be considered as the most generic ones among all Moishezon twistor spaces on $4\mathbb{CP}^2$. The most characteristic property of these twistor spaces is that they have a natural structure of double covering over a very simple rational threefold, so called a scroll of planes over a conic. Moreover, the branch divisor of the covering is always a cut of the scroll by a *quartic hypersurface*. This allowed us to describe each of these twistor spaces by a single quartic polynomial. It was shown that this quartic polynomial takes not general but a very special form. This description has a clear advantage when describing a global structure of the moduli space of these twistor spaces.

While these twistor spaces are most generic among all Moishezon twistor spaces on $4\mathbb{CP}^2$ as above, we showed in another article [5] (in which we classified all Moishezon twistor spaces on $4\mathbb{CP}^2$) that there still exist other three families of twistor spaces on $4\mathbb{CP}^2$ which have a natural structure of a double covering over the same scroll. The twistor spaces in these families are strictly different from the ones in [4], but they are obtained from the generic ones by taking a limit under deformations. In particular, the branch divisor of the double covering is still a cut of the scroll by a quartic hypersurface, while its defining equation should be subject to stronger constraint.

Among the three families, the most special one is nothing but the family of twistor spaces studied in [3] (specialized to the case of $4\mathbb{CP}^2$). In particular for twistor spaces in this family the defining equation of the quartic hypersurface is already determined. The main purpose of this article is to obtain the equation of the quartic hypersurface for the remaining two families of the spaces

As explained above, twistor spaces whose anticanonical map is of degree two can be classified into four types. Distinction of the types can be detected by the number of irreducible components of the base curve for the half-anticanonical system on the twistor spaces. We just call these as type I, II, III and IV (Definition 2.2; this naming is justified when we obtain defining equation of the quartic hypersurface.). Type I is most generic (so they are studied in [4]) and type IV is most special (so they are studied in [3]). Hence in this paper we study those of type II and type III. In Section 2 we take a general member of the half-anticanonical system and investigate structure of its bi-anticanonical system. The last system exhibits the surface as a double covering of \mathbb{CP}^2 with branch being a quartic curve, whose singularities depend on the types I–IV (as displayed in Figure 1).

As above, the structure of the twistor spaces can be captured through the anticanonical system. In Section 3, we find *reducible* anti-canonical divisor(s) of the twistor space, which will be a key for determining the equation of the branch divisor of the anticanonical map.

In contrast with the generic ones studied in [4], the base locus of the anticanonical map of the present twistor spaces is quite complicated. In Section 4 we give a full elimination of this locus through explicit blowups. In particular, we find that the anticanonical map contracts some divisors to curves. (For the case of type I such a divisor does not exist as investigated in [4].) This information will give a strong constraint for the equation of the branch divisor of the anticanonical map.

In Section 5, by using the reducible anticanonical divisor(s) found in Section 3, we first find five special hyperplanes in \mathbb{CP}^4 whose intersection with the scroll touches the branch divisor along a curve. These curves are called double curves. Next we show that there exists a hyperquadric in \mathbb{CP}^4 which contains all these five double curves. Finally by using the hyperquadric and also the information about the anticanonical map obtained in Section 4, we determine defining equation of the branch divisor of the anticanonical map, for both cases of type II and type III (Theorem 5.3). The argument in the proof is mostly algebraic. We also give an account about how the defining equation of the quartic hypersurface degenerates when the twistor space changes the type.

In Section 6 we compute dimension of the moduli spaces of these twistor spaces. The conclusion including the cases of type I and type IV is as follows:

	type I	type II	type III	type IV
dimension of the moduli space	9	7	5	4
dimension of the automorphism group	0	0	0	1

Thus whole picture is now understood to a considerable degree.

Finally, as mentioned in [4], it looks quite certain that the twistor spaces which have the structure of the double covering over the scroll can be generalized to the ones over $n\mathbb{CP}^2$, n being arbitrary. The results in [3] mean this is actually the case for type IV spaces. However, after writing [4], the author noticed that *the twistor spaces on $4\mathbb{CP}^2$ studied in [4] (i.e. type I spaces) cannot be generalized to $n\mathbb{CP}^2$, as long as we stick to the linear system $|(n-2)K^{-1/2}|$* . On the other hand, for those of type II and type III, there seems to be a good chance for generalization by using $|(n-2)K^{-1/2}|$, like [3]. This is a reason why we study these cases in depth.

Notations. For a twistor space, the natural square root of the anticanonical bundle is denoted by F . (Hence $2F$ is the anticanonical bundle.) The dimension of a linear system always means the dimension of the parameter projective space. For a line bundle $L \rightarrow X$, we write $h^i(X, L) = \dim H^i(X, L)$. For $s \in H^0(X, L)$ with $s \neq 0$, we denote (s) for the zero-divisor of s . For a curve C and a divisor D on X , the intersection number of C and D is denoted by $(C, D)_X$ or just (C, D) .

2. THE ANTICANONICAL SYSTEM OF THE TWISTOR SPACES

We first make it clear which twistor spaces on $4\mathbb{CP}^2$ we are going to investigate. For this we recall the following result which is one of a consequence from the classification of all Moishezon twistor spaces, obtained in [5]:

Proposition 2.1. ([5, Theorem 1.1]) *Let Z be a Moishezon twistor space on $4\mathbb{CP}^2$ and suppose that the anticanonical map $\Phi = \Phi_{|2F|}$ is (rationally) of degree two over the image. Then we have the following. (i) $\dim |2F| = 4$ and the image $\Phi(Z) \subset \mathbb{CP}^4$ is a scroll of planes over a conic, (ii) $\dim |F| = 1$ and $\text{Bs } |F|$ is a cycle of smooth rational curves, (iii) the cycle consists of 4, 6, 8 or 10 irreducible components.*

Here, by *the scroll of planes over a conic*, we mean the inverse image $\pi^{-1}(\Lambda)$, where $\pi : \mathbb{CP}^4 \rightarrow \mathbb{CP}^2$ is a linear projection and Λ is a conic in \mathbb{CP}^2 . Namely, the scroll is a union of all planes in \mathbb{CP}^4 which contain a fixed line. Clearly such a scroll is unique up to projective transformations of \mathbb{CP}^4 .

An obvious relation between $|F|$ and $|2F|$ gives the following commutative diagram

$$(2.1) \quad \begin{array}{ccc} Z & \xrightarrow{\Phi_{|2F|}} & \mathbb{CP}^4 \\ \Phi_{|F|} \downarrow & & \downarrow \pi \\ \mathbb{CP}^1 & \longrightarrow & \mathbb{CP}^2. \end{array}$$

where the bottom arrow is an embedding of \mathbb{CP}^1 onto the conic $\Lambda \subset \mathbb{CP}^2$.

Throughout this paper we denote the cycle $\text{Bs } |F|$ appeared in (ii) of Proposition 2.1 by the letter C (as in [5]). Of course, this cycle C is itself real. The number of its irreducible components is significant because it is directly connected with the structure of the branch divisor of the degree two rational map Φ (over the scroll) in (i) of the proposition. So we introduce the following

Definition 2.2. Let Z be as in Proposition 2.1. Then according to the number 4, 6, 8 or 10 of the irreducible components of the cycle C , we call Z is of *type I, II, III or IV* respectively.

We note that the twistor spaces studied in [4] are exactly those of type I, and that the twistor spaces studied in [3] are those of type IV if we substitute $n = 4$ in the paper. In particular in these papers a defining equation of the branch divisor on the scroll is explicitly obtained. So in this article we are concerned with the cases of type II and type III. We also mention that among these four kinds of the twistor spaces, type I is most generic and type IV is most special in the sense that, if $I \leq A < B \leq IV$, then type A is obtained from type B by small deformation; in other words, type B is obtained as a limit through a family of type A twistor spaces. So it might be possible to say that the present twistor spaces are intermediately degenerate ones among all twistor spaces (on $4\mathbb{CP}^2$) whose anticanonical map is of degree two.

Let Z be as in Proposition 2.1 and $S \in |F|$ any real irreducible member, which is always smooth rational surface with $K_S^2 = 0$ by [8]. As $\dim |F| = 1$, we have $\dim |K_S^{-1}| = 0$, and the unique anticanonical curve is exactly the cycle C . By reality we can write it as

$$(2.2) \quad C = \sum_{i=1}^k C_i + \sum_{i=1}^k \overline{C}_i,$$

where $k = 2, 3, 4, 5$ according to type I, II, III, IV respectively. Here we are taking the numbering for the components in a natural way that C_i and C_{i+1} intersect for $1 \leq i \leq k-1$ and C_k and \overline{C}_1 intersect. Then by [5] the sequence obtained by arranging the self-intersection numbers of the components is respectively given as follows (after a proper cyclic

permutation and an exchange of orientation):

$$(2.3) \quad -3, -1, -3, -1 \quad \text{for type I,}$$

$$(2.4) \quad -3, -2, -1, -3, -2, -1 \quad \text{for type II,}$$

$$(2.5) \quad -3, -2, -2, -1, -3, -2, -2, -1 \quad \text{for type III,}$$

$$(2.6) \quad -3, -2, -2, -2, -1, -3, -2, -2, -2, -1 \quad \text{for type IV.}$$

These indicate that, for example in the case of type II, the self-intersection numbers of the components $C_1, C_2, C_3, \overline{C}_1, \overline{C}_2, \overline{C}_3$ in S are $-3, -2, -1, -3, -2, -1$ respectively. As we will see in Section 4, these numbers are directly related with birational geometry of the twistor spaces.

From the relation $2F|_S \simeq 2K_S^{-1}$, the structure of the anticanonical map of the twistor spaces may be investigated via the bi-anticanonical system of the surface S . For type I (resp. type IV) twistor spaces the last system is investigated in [4, Section 2] (resp. [3, Section 2.2]; substitute $n = 4$). We now write down the corresponding properties for the cases of type II and type III respectively.

Proposition 2.3. *For the case of type II, the bi-anticanonical system of S satisfies the following: (i) the fixed component is $C_1 + C_2 + \overline{C}_1 + \overline{C}_2$, (ii) if we remove this fixed component, the resulting system is base point free and 2-dimensional, (iii) if $\phi : S \rightarrow \mathbb{CP}^2$ is the induced morphism, ϕ is of degree two, and the branch divisor is a quartic curve which has two ordinary nodes, (iv) the morphism ϕ maps the connected curves $C_3 \cup \overline{C}_1$ and $\overline{C}_3 \cup C_1$ to the two nodes, (v) ϕ maps both of the curves C_2 and \overline{C}_2 to the line l connecting the two nodes, (vi) $\phi^{-1}(l) = C$.*

Proposition 2.4. *For the case of type III, the bi-anticanonical system of S satisfies the following: (i) the fixed component is $C_1 + C_2 + C_3 + \overline{C}_1 + \overline{C}_2 + \overline{C}_3$, (ii) after removing this, the resulting system is base point free and 2-dimensional, (iii) the induced morphism $\phi : S \rightarrow \mathbb{CP}^2$ is of degree two, and the branch divisor is a quartic curve with two cusps, (iv) ϕ maps the connected curves $\overline{C}_4 \cup C_1 \cup C_2$ and $C_4 \cup \overline{C}_1 \cup \overline{C}_2$ to the two cusps, (v) ϕ maps both of the curves C_3 and \overline{C}_3 to the line l connecting the two cusps, (vi) $\phi^{-1}(l) = C$.*

We omit proofs of these two propositions but instead illustrate the branch quartic curve and the line l as in Figure 1. (In the figure the branch quartic curves in the cases of type I and type IV are included from coherency of these four types of the spaces.)

Going back to the twistor spaces, as consequences of these two propositions, we have

Proposition 2.5. *(i) For the case of type II, we have $\text{Bs } |2F| = C_1 \cup C_2 \cup \overline{C}_1 \cup \overline{C}_2$. (ii) For the case of type III, we have $\text{Bs } |2F| = C_1 \cup C_2 \cup C_3 \cup \overline{C}_1 \cup \overline{C}_2 \cup \overline{C}_3$.*

Proof. These immediately follow from (i) of Propositions 2.3 and 2.4, and surjectivity of the restriction map $H^0(Z, 2F) \rightarrow H^0(S, 2K_S^{-1})$ which was proved in [5, Proposition 2.10]. (Note that the last surjectivity may be shown very readily since we are on $4\mathbb{CP}^2$.) \square

We end this section by the following property about reducible members of the pencil $|F|$, which seems to have been well-understood (see Kreussler [6]).

Proposition 2.6. *Let Z be a twistor space on $n\mathbb{CP}^2$ satisfying $\dim |F| = 1$ and suppose that $\text{Bs } |F|$ is a cycle of rational curves, written still by C . Then the number of reducible members of the pencil $|F|$ is equal to the half of the number of the components of C . Further, all the reducible members are of the form $S_i^+ + S_i^-$, where S_i^\pm are degree-one divisors satisfying*

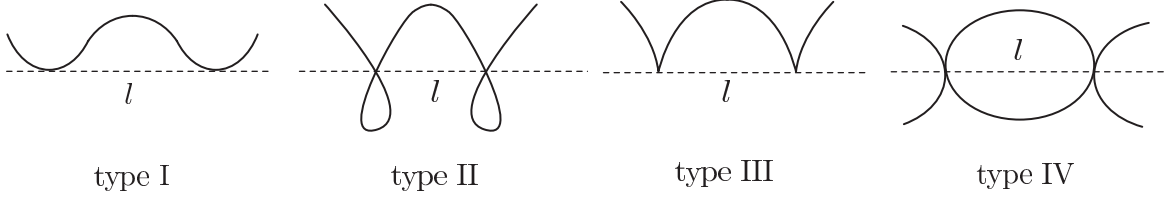


FIGURE 1. The branch quartic curves, in relation with the special line l .

$\overline{S}_i^+ = S_i^-$. Furthermore, each of these members divides the cycle C into ‘halves’ in the sense that both $S_i^+ \cap C$ and $S_i^- \cap C$ are connected and have equal components.

In particular, Z of type II (resp. III) has exactly three (resp. four) reducible members of the pencil $|F|$. We fix the indices of the reducible members by the property that $S_i^+ \cap S_i^-$ (which is always a twistor line) goes through the point $C_i \cap C_{i+1}$, where we read $C_{i+1} = \overline{C}_1$ when C_{i+1} does not exist. Also we make distinction between S_i^+ and S_i^- by the property that $S_i^+ \supset \overline{C}_1$. These degree one divisors will play some role in our analysis of the twistor spaces.

3. EXISTENCE OF SOME REDUCIBLE ANTICANONICAL DIVISORS

In this section we prove that on the twistor spaces under consideration there exist some *irreducible* degree two divisors, which do not belong to the fundamental system $|F|$. Similarly to the case of type I studied in [4], these divisors will be a key for obtaining the defining equation of the branch divisor of the anticanonical map. However in the present case for finding these divisors we need different computations from the case of type I given in [4].

Let $S^2H^0(F)$ be the subspace in $H^0(2F)$ generated by all elements in $H^0(F)$. This is a 3-dimensional subspace of $H^0(2F)$.

Proposition 3.1. *Let Z be a Moishezon twistor space on $4\mathbb{CP}^2$ whose anticanonical map is (rationally) of degree 2 over the image. (i) If Z is of type II, then there exist distinct two anticanonical divisors $X_1 + \overline{X}_1$ and $X_2 + \overline{X}_2$ on Z which do not belong to the subsystem $|S^2H^0(F)|$. (ii) If Z is of type III, there exists an anticanonical divisor $X + \overline{X}$ on Z which does not belong to the subsystem $|S^2H^0(F)|$.*

We remark that $X_i + \overline{X}_i \notin |S^2H^0(F)|$ implies irreducibility of X_i and \overline{X}_i . We mention that if Z is of type I (resp. type IV) there exist *three* (resp. *no*) such anticanonical divisors as shown in [4, Proposition 4.1] (resp. [3, Theorem 5.1, Eq. (100)]). This difference for the number of such divisors will be reflected to a form of the defining equation of the branch divisor of the anticanonical map.

For the proof of Proposition 3.1, we need the following

Lemma 3.2. *Let Z be as in Proposition 3.1 and $S \in |F|$ a real irreducible member. Then regardless of the type, there is a birational morphism $\epsilon : S \rightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$ preserving the real structure, such that the image $\epsilon(C)$ is an anticanonical curve on $\mathbb{CP}^1 \times \mathbb{CP}^1$, and such that $\epsilon(C_1), \epsilon(\overline{C}_1) \in |\mathcal{O}(1, 0)|$ and $\epsilon(C_2), \epsilon(\overline{C}_2) \in |\mathcal{O}(0, 1)|$.*

As a proof of this lemma is somewhat tedious to write down, we just mention that it suffices to notice that if A is any anticanonical curve on a non-singular surface and D is a (-1) -curve satisfying $D \not\subset A$, then A must intersect D at a unique smooth point of A and the intersection is transversal.

Proof of Proposition 3.1 (i). Let $\epsilon : S \rightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$ be the birational morphism as in Lemma 3.2. This canonically determines a set of effective curves $\{e_i, \bar{e}_i \mid 1 \leq i \leq 4\}$ satisfying $(e_i, e_j)_S = -\delta_{ij}$ and $(e_i, \bar{e}_j)_S = 0$ for any i and j . (Since ϵ might involve a blowup at an infinitely near point, e_i and \bar{e}_i can be reducible in general.) From the self-intersection numbers of the components of the cycle C in the case of type II, we can suppose that

$$(3.1) \quad e_3 = \bar{C}_3, \quad (e_1, C_1) = (e_2, C_1) = 1, \quad (e_4, C_2) = 1.$$

Next let $t : Z \rightarrow 4\mathbb{CP}^2$ be the twistor fibration map, and let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be orthonormal basis of $H^2(4\mathbb{CP}^2, \mathbb{Z})$ which are uniquely determined by the property that $(t^*\alpha_i)|_S = e_i - \bar{e}_i$ in $H^2(S, \mathbb{Z})$.

We first show that $h^0(F - t^*\alpha_1|_S) = 1$. In the following for simplicity we write α_i for the lift $t^*\alpha_i$, and also write $(a, b) := \epsilon^*\mathcal{O}(a, b) \in H^2(S, \mathbb{Z})$. Then from (3.1) and the bidegrees in Lemma 3.2 we have the following relations in $H^2(S, \mathbb{Z})$:

$$\begin{aligned} C_1 &= (1, 0) - e_1 - e_2 - e_3, \\ C_2 &= (0, 1) - \bar{e}_3 - e_4. \end{aligned}$$

By using this, we compute as

$$\begin{aligned} (F - \alpha_1, C_1)_Z &= (K_S^{-1} - (e_1 - \bar{e}_1), (1, 0) - e_1 - e_2 - e_3)_S \\ &= ((2, 2) - 2e_1 - e_2 - e_3 - e_4 - \bar{e}_2 - \bar{e}_3 - \bar{e}_4, (1, 0) - e_1 - e_2 - e_3)_S \\ (3.2) \quad &= 2 + (-2 - 1 - 1) = -2. \end{aligned}$$

Hence the curve C_1 is a fixed component of the system $|(F - \alpha_1)|_S|$. In a similar way we further find $((F - \alpha_1)|_S - C_1, C_2)_S = -1$, meaning C_2 is also a fixed component of the same system. We then have

$$(3.3) \quad (F - \alpha_1)|_S - C_1 - C_2 = (1, 1) - e_1 - \bar{e}_2 - \bar{e}_4.$$

Now, since the cycle C on S consists of exactly 6 components, for $i = 1, 2$, the points $\epsilon(e_i)$ and $\epsilon(\bar{e}_i)$ respectively belong to $\epsilon(C_1)$ and $\epsilon(\bar{C}_1)$, which are not the singular points of the cycle $\epsilon(C)$. By the same reason, $\epsilon(e_4)$ and $\epsilon(\bar{e}_4)$ are points on $\epsilon(C_2)$ and $\epsilon(\bar{C}_2)$ respectively, which are not the singular points of the cycle $\epsilon(C)$. From these it follows that there exists a unique member of the linear system of (3.3), and that the member does not contain any of the irreducible components of the cycle C . (Such a member is exactly the strict transform of the $(1, 1)$ -curve going through the three points $\epsilon(e_1), \epsilon(\bar{e}_2)$ and $\epsilon(\bar{e}_4)$.) Thus we get $h^0(F - \alpha_1|_S) = 1$, as claimed. In the same manner, we obtain $(F - \alpha_2|_S, C_1)_S < 0$, $((F - \alpha_2)|_S - C_1, C_2)_S < 0$ and $(F - \alpha_2)|_S - C_1 - C_2 = (1, 1) - \bar{e}_1 - e_2 - \bar{e}_4$, which again imply $h^0(F - \alpha_2|_S) = 1$ and that the unique member of $|F - \alpha_2|_S|$ does not contain any of the irreducible components of the cycle C .

Next let $s \in H^0(F)$ be an element such that $(s) = S$, and for $i = 1, 2$ we consider the obvious exact sequence

$$(3.4) \quad 0 \longrightarrow \mathcal{O}_Z(-\alpha_i) \xrightarrow{\otimes s} F \otimes \mathcal{O}_Z(-\alpha_i) \longrightarrow (F - \alpha_i)|_S \longrightarrow 0.$$

By Riemann-Roch formula we have $\chi(\mathcal{O}_Z(-\alpha_i)) = 0$ and by Hitchin vanishing [2] we have $H^2(\mathcal{O}_Z(-\alpha_i)) = 0$. Also $H^0(\mathcal{O}_Z(-\alpha_i)) = H^3(\mathcal{O}_Z(-\alpha_i)) = 0$ by trivial reason. Hence we obtain $H^1(\mathcal{O}_Z(-\alpha_i)) = 0$. Thus by the cohomology exact sequence of (3.4) we obtain $h^0(F - \alpha_i) = h^0(F - \alpha_i|_S)$. Therefore we obtain $h^0(F - \alpha_i) = 1$ for $i = 1, 2$.

Let $x_i \in H^0(F - \alpha_i)$ be a non-zero element for $i = 1, 2$. Then $\bar{x}_i := \overline{\sigma^*x_i}$, where σ is the real structure of Z , is a non-zero element of $H^0(F + \alpha_i)$. Hence the product $x_i\bar{x}_i$ belongs

to $H^0(2F)$. For finishing a proof we have to show $x_i\bar{x}_i \notin S^2H^0(F)$. For this, we recall from the above argument that the divisor $(x_1|_S)$ contains the unique curve of the system (3.3), and that the curve does not contain components of the cycle C . In particular the curve $(x_1\bar{x}_1|_S)$ is not contained in C . On the other hand, any $x \in S^2H^0(F)$ clearly satisfies $x|_S = 0$ or $(x|_S) = 2C$. Thus we conclude $x_1\bar{x}_1 \notin S^2H^0(F)$. By the same argument we also obtain $x_2\bar{x}_2 \notin S^2H^0(F)$. Hence by letting $X_i = (x_i)$ and $\bar{X}_i = (\bar{x}_i)$ for $i = 1, 2$, we finish a proof of Proposition 3.1 (i). \square

The idea for (ii) being similar but again subtle, we give an outline:

Proof of Proposition 3.1 (ii). Let $\epsilon : S \rightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$, $\{e_i, \bar{e}_i \mid 1 \leq i \leq 4\}$ and $\{\alpha_i \mid 1 \leq i \leq 4\}$ have the same meaning as in (i). Then this time we can suppose

$$(3.5) \quad C_1 = (1, 0) - e_1 - e_2 - e_3, \quad C_2 = (0, 1) - \bar{e}_3 - e_4, \quad C_3 = \bar{e}_3 - \bar{e}_2, \quad C_4 = \bar{e}_2$$

in $H^2(S, \mathbb{Z})$. By computing intersection numbers we can show that $|(F - \alpha_1)|_S|$ has $C_1 + C_2 + C_3$ as fixed components, and that $(F - \alpha_1)|_S - C_1 - C_2 - C_3 = (1, 1) - e_1 - \bar{e}_3 - \bar{e}_4$. Further this system has a unique member, which is irreducible. Then by the same argument for (i) we deduce $|F - \alpha_1|$ has a unique member X , and that $X + \bar{X} \in |2F|$ and $X + \bar{X} \notin |S^2H^0(F)|$. \square

4. ANALYSIS OF THE ANTICANONICAL SYSTEM OF THE TWISTOR SPACES

In this section we analyze structure of the anticanonical map of the twistor spaces. For type I twistor spaces, as showed in [4, Proposition 3.2], the base locus of the anticanonical system $|2F|$ can be eliminated by just blowing up the two (-3) -curves in the cycle C (see (2.3)). However, this is never the case for type II and type III. In this section we explicitly provide a full elimination of the base locus $\text{Bs}|2F|$ for these two cases. This is a core part of our analysis, and indispensable for obtaining a constraint for the defining equation of the branch divisor of the anticanonical map. (In this section we do not need the results in the previous section.)

The elimination we take here is different from [4] for the case of type I, in that we first blowup the whole of the cycle C , not the base curves themselves. (So it is not a ‘minimal’ elimination.) Though this yields a lot of ordinary double points, this provides Z a structure of fibration, and this makes much easier to keep track of the base locus of the linear system, which is otherwise quite difficult.

4.1. The case of type II. Let Z be a twistor space of $4\mathbb{CP}^2$ whose anticanonical map is degree two, which is of type II. As before let $\Phi : Z \rightarrow \mathbb{CP}^4$ be the anticanonical map, $S \in |F|$ a real irreducible member, and C the unique anticanonical curve of S (i.e. $C = \text{Bs}|F|$). C is a cycle of six rational curves. Let $f : Z \rightarrow \mathbb{CP}^1$ be the rational map associated with the pencil $|F|$. The last \mathbb{CP}^1 can be naturally identified with the conic Λ through the diagram (2.1). The map f has the cycle C as the indeterminacy locus. Let $\mu_1 : Z_1 \rightarrow Z$ be the blowing-up at C , and E_i and \bar{E}_i ($1 \leq i \leq 3$) the exceptional divisors over C_i and \bar{C}_i respectively. Then thanks to the fact that C is a cycle, all these exceptional divisors are isomorphic to $\mathbb{CP}^1 \times \mathbb{CP}^1$. Write the composition $Z_1 \xrightarrow{\mu_1} Z \xrightarrow{f} \mathbb{CP}^1$ by f_1 . Then since $F|_S \simeq \mathcal{O}_S(C)$, $f_1 : Z_1 \rightarrow \mathbb{CP}^1$ is a morphism. By μ_1 any fiber of f_1 can be biholomorphically identified with a member of the pencil $|F|$. Hence by Proposition 2.6, f_1 has exactly three reducible fibers. Let $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{CP}^1$ be the points such that $f^{-1}(\lambda_i) = S_i^+ \cup S_i^-$. We put $L_i = S_i^+ \cap S_i^-$ for $1 \leq i \leq 3$. These are twistor lines.

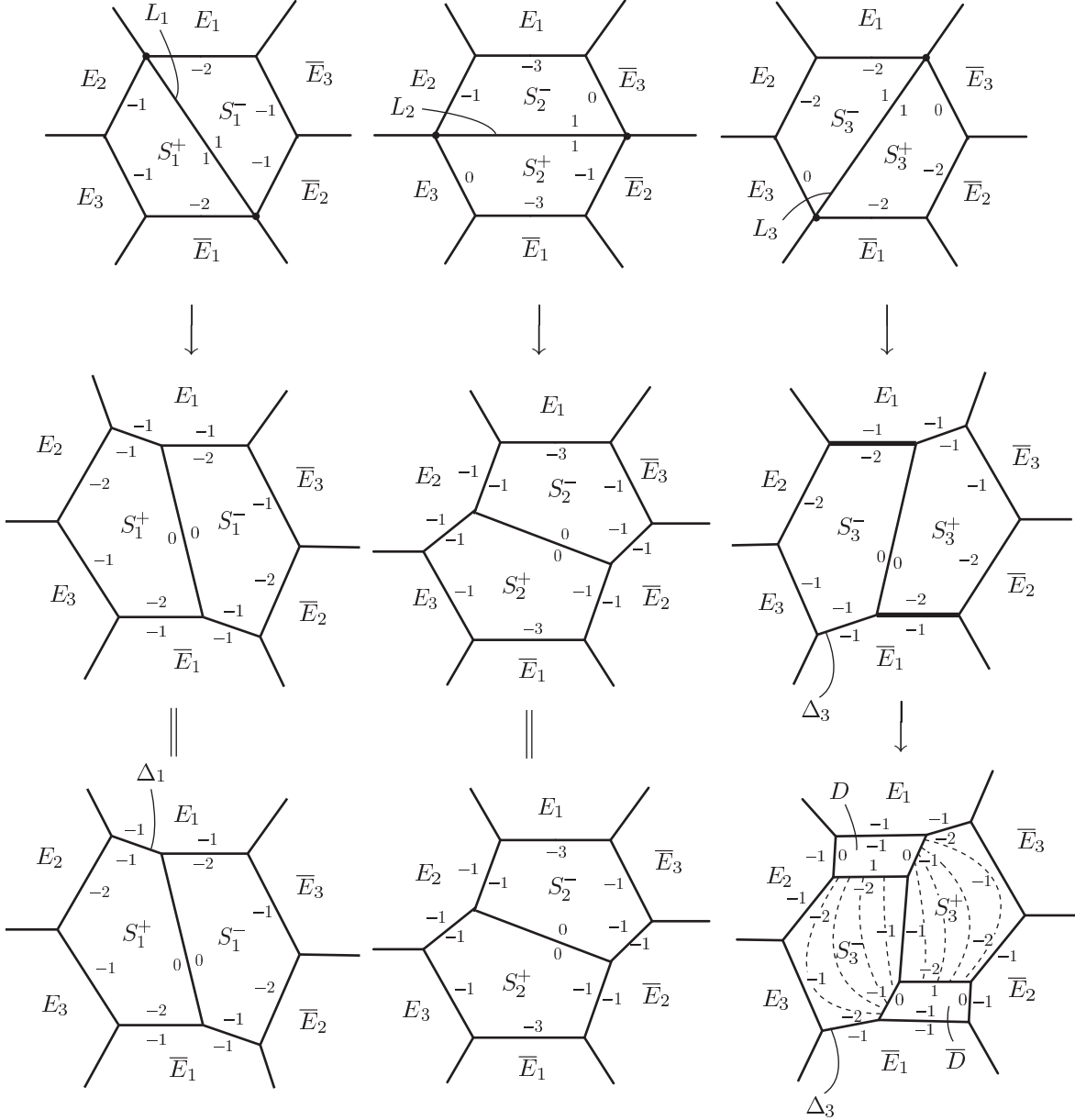


FIGURE 2. The blowups for the case of type II. The first, second, and the third rows are contained in Z_1 , Z_2 and Z_3 respectively.

For simplicity we use the same letters to denote divisors and twistor lines in the original space Z and their strict transforms in Z_1 . In the first row of Figure 2 we illustrate Z_1 in a neighborhood of each reducible fiber $(S_i^+ \cup S_i^-)$ of f_1 . As a computation using local coordinates shows, Z_1 has an ordinary node at the point where four faces meet, and these are indicated as dotted points. (On each reducible fiber there are two such points.) We take small resolutions for these six nodes as displayed in the second row of Figure 2. This inserts two \mathbb{CP}^1 -s in each reducible fiber of f_1 , and does not change any other part. Let Z_2 be the resulting non-singular space, and write $\mu_2 : Z_2 \rightarrow Z$ and $f_2 : Z_2 \rightarrow \mathbb{CP}^1$ for the

compositions $Z_2 \rightarrow Z_1 \xrightarrow{\mu_1} Z$ and $Z_2 \rightarrow Z_1 \xrightarrow{f_1} \mathbb{CP}^1$ respectively. We again use the same letter for strict transforms into Z_2 . Then because of Proposition 2.3 (i), the pullback system $|\mu_2^*(2F)|$ has $E_1 + E_2 + \overline{E}_1 + \overline{E}_2$ as fixed components at least. Hence we put

$$\mathcal{L}_2 := \mu_2^*(2F) - (E_1 + E_2 + \overline{E}_1 + \overline{E}_2).$$

Once we fix a section v of the line bundle $\mathcal{O}_{Z_2}(E_1 + E_2 + \overline{E}_1 + \overline{E}_2)$, the birational morphism $\mu_2 : Z_2 \rightarrow Z$ naturally induces a isomorphism $H^0(2F) \simeq H^0(\mathcal{L}_2)$. Moreover noticing the basic relation

$$(4.1) \quad f_2^* \mathcal{O}_{\mathbb{CP}^1}(1) \simeq \mu_2^* F - \sum_{1 \leq i \leq 3} (E_i + \overline{E}_i),$$

we obtain

$$(4.2) \quad \mathcal{L}_2 \simeq f_2^* \mathcal{O}(2) + E_1 + E_2 + 2E_3 + \overline{E}_1 + \overline{E}_2 + 2\overline{E}_3.$$

Therefore for any fiber $S_\lambda := f_2^{-1}(\lambda)$, we have $\mathcal{L}_2|_{S_\lambda} \simeq \mathcal{O}_{S_\lambda}(E_1 + E_2 + 2E_3 + \overline{E}_1 + \overline{E}_2 + 2\overline{E}_3)$. This is useful for computing the base locus of $|\mathcal{L}_2|$. In particular if S is a non-singular member of the pencil $|F|$, this isomorphism identifies $|\mathcal{L}_2|_S$ with the system $|C_1 + C_2 + 2C_3 + \overline{C}_1 + \overline{C}_2 + 2\overline{C}_3|$, which is the movable part of $|2K_S^{-1}|$ by Proposition 2.3 (ii).

Lemma 4.1. *The linear system $|\mathcal{L}_2|$ on Z_2 has the following properties: (i) $|\mathcal{L}_2|$ has no fixed component, (ii) the two smooth rational curves $(S_3^- \cap E_1)$ and $(S_3^+ \cap \overline{E}_1)$ are base curves of $|\mathcal{L}_2|$. (In Figure 2, these are displayed as bold segments. Also, at this stage we do not prove these are all base points of $|\mathcal{L}_2|$, although this is actually the case.)*

Proof. For (i), by Proposition 2.3 (i), it is enough to show that the exceptional divisors E_i and \overline{E}_i , $i = 1, 2$, are not a fixed component of $|\mathcal{L}_2|$. For any smooth fiber S of f_2 we have the following commutative diagram:

$$\begin{array}{ccc} H^0(Z_2, \mathcal{L}_2) & \xrightarrow{\sim} & H^0(Z, 2F) \\ \downarrow & & \downarrow \\ H^0(S, \mathcal{L}_2|_S) & \xrightarrow{\sim} & H^0(S, 2K^{-1}), \end{array}$$

where the upper horizontal isomorphism is the composition of a multiplication of the section v and the isomorphism $H^0(Z_2, \mu_2^*(2F)) \simeq H^0(Z, 2F)$, and the lower horizontal arrow is its restriction to S , which is also isomorphic by (4.2) and Proposition 2.3. Further, the right vertical arrow is surjective ([5, Proposition 2.10]). Hence so is the left arrow. Therefore, since the linear system $|\mathcal{L}_2|_S \simeq |C_1 + C_2 + 2C_3 + \overline{C}_1 + \overline{C}_2 + 2\overline{C}_3|$ is base point free by Proposition 2.3 (ii), we obtain that $|\mathcal{L}_2|$ has no fixed point on the smooth fiber S . Hence any E_i and \overline{E}_i cannot be a fixed component of $|\mathcal{L}_2|$.

For (ii), we temporary write $C_i := S_3^- \cap E_i (\subset Z_2)$ for $1 \leq i \leq 3$, and also let $\Delta_3 = S_3^- \cap \overline{E}_1$ (which is an exceptional curve of the small resolution $Z_2 \rightarrow Z_1$). Then by (4.2), we compute

$$\begin{aligned} (\mathcal{L}_2, C_1)_{Z_2} &= (E_1 + E_2 + 2E_3 + \overline{E}_1 + \overline{E}_2 + 2\overline{E}_3, C_1)_{Z_2} \\ &= (C_1 + C_2 + 2C_3 + \Delta_3, C_1)_{S_3^-} \\ &= -2 + 1 = -1. \end{aligned}$$

Hence C_1 is a base curve of $|\mathcal{L}_2|$. By reality $\overline{C}_1 = S_3^+ \cap \overline{E}_1$ is also a base curve of $|\mathcal{L}_2|$. \square

By Lemma 4.1, we let $\mu_3 : Z_3 \rightarrow Z_2$ be the blowup at the base curves $(S_3^- \cap E_1) \cup (S_3^+ \cap \overline{E}_1)$ and D and \overline{D} the exceptional divisors respectively. (See the lower right in Figure 2.) D and \overline{D} are isomorphic to $\mathbb{P}(\mathcal{O}(1) \oplus \mathcal{O})$ over \mathbb{CP}^1 . We put

$$\mathcal{L}_3 := \mu_3^* \mathcal{L}_2 - (D + \overline{D}).$$

The blowup μ_3 induces an isomorphism $H^0(\mathcal{L}_3) \simeq H^0(\mathcal{L}_2)$. Let $f_3 : Z_3 \xrightarrow{\mu_3} Z_2 \xrightarrow{f_2} \mathbb{CP}^1$ be the composition. Then the relation (4.2) is valid for \mathcal{L}_3 if we replace f_2 with f_3 . (But note that on Z_3 the surfaces S_3^- and E_1 are separated by D as in Figure 2.) The next proposition implies that the blowup μ_3 terminates an elimination of the base locus of the original system $|2F|$ on the twistor space.

Proposition 4.2. *The system $|\mathcal{L}_3|$ on Z_3 is base point free.*

Proof. Again we use the same letter to mean the strict transforms into Z_3 . By computing the degrees of the line bundle \mathcal{L}_3 over the curves on E_1 and E_3 seen in Figure 2, it is possible to show that the restrictions of \mathcal{L}_3 to E_1 and E_3 are both *trivial*. On the other hand, the system $|\mathcal{L}_3|$ also does not have any E_i or \overline{E}_i as a fixed component by Lemma 4.1 (i). These imply that E_1 and E_3 must be disjoint from $\text{Bs } |\mathcal{L}_3|$. In particular, with the aid of Proposition 2.5 (i), we have

$$\text{Bs } |\mathcal{L}_3| \subset E_2 \cup \overline{E}_2 \cup D \cup \overline{D}.$$

Next we see that $(\text{Bs } |\mathcal{L}_3|) \cap E_2 = \emptyset$. Since $|\mathcal{L}_2|$ has no fixed point on any smooth fiber of f_2 as shown in the proof of Lemma 4.1, the same is true for $|\mathcal{L}_3|$ for any smooth fiber of f_3 . On the other hand, we can deduce that the system $|\mathcal{L}_3|_{E_2}|$ is a pencil without a base point, whose members are sections of the natural projection $E_2 \rightarrow \Lambda$ (which is the restriction of $f_3 : Z_3 \rightarrow \mathbb{CP}^1$). These imply that $|\mathcal{L}_3|$ does not have a base point on E_2 too.

Finally we see that $(\text{Bs } |\mathcal{L}_3|) \cap D = \emptyset$. For this we first notice from (4.2) and Figure 2 that the restriction $\mathcal{L}_3|_D$ is isomorphic to a pullback of $\mathcal{O}(1)$ by the blowdown $D \rightarrow \mathbb{CP}^2$. Therefore in order to show $(\text{Bs } |\mathcal{L}_3|) \cap D = \emptyset$, it suffices to prove that the rational map Φ_3 associated to $|\mathcal{L}_3|$ satisfies $\dim \Phi_3(D) = 2$. We show this by proving that $\Phi_3(D)$ contains two distinct lines.

From the construction Φ_3 factors as $Z_3 \rightarrow Z \xrightarrow{\Phi} \mathbb{CP}^4$, where $Z_3 \rightarrow Z$ is the composition of all blowups. Therefore the image of Φ_3 is the scroll $Y = \pi^{-1}(\Lambda)$. Moreover each fiber of the fibration $f_3 : Z_3 \rightarrow \mathbb{CP}^1$ is mapped (by Φ_3) to a fiber plane of the scroll $Y \rightarrow \Lambda$ (see the diagram (2.1)). Next we show that $\Phi_3(S_3^-)$ is a line. By (4.2), temporarily writing $C_i = S_3^- \cap E_i$ for $1 \leq i \leq 3$ on Z_3 and recalling that we have subtracted the divisor D when defined \mathcal{L}_3 , we have

$$(4.3) \quad \mathcal{L}_3|_{S_3^-} \simeq \mathcal{O}_{S_3^-}(C_2 + 2C_3 + \Delta_3).$$

From this it is easy to see that the system $|\mathcal{L}_3|_{S_3^-}|$ is a pencil without a base point, and the associated morphism $S_3^- \rightarrow \mathbb{CP}^1$ has the curves $D \cap S_3^-$ and $\overline{D} \cap S_3^-$ as sections. (In the lower right of Figure 2, fibers of the morphism $S_3^- \rightarrow \mathbb{CP}^1$ are indicated as dotted curves.) In particular, if the restriction $H^0(Z_3, \mathcal{L}_3) \rightarrow H^0(S_3^-, \mathcal{L}_3|_{S_3^-}) \simeq \mathbb{C}^2$ is not surjective, the system $|\mathcal{L}_3|$ has base points along a fiber of the morphism $S_3^- \rightarrow \mathbb{CP}^1$. But this cannot happen since we already know $\text{Bs } |\mathcal{L}_3| \subset D \cup \overline{D}$. Therefore the above restriction is surjective. This implies that $\text{Bs } |\mathcal{L}_3| \cap S_3^- = \emptyset$ and $\Phi_3(S_3^-)$ is a line. Hence $\Phi_3(S_3^- \cap D)$ is also the

same line. In particular $\Phi_3(D)$ contains the line $\Phi_3(S_3^-)$. Also, this line can be written as $\Phi_3(S_3^- \cap \overline{D})$ (since $S_3^- \cap \overline{D}$ was a section of the morphism $S_3^- \rightarrow \mathbb{CP}^1$).

By the real structure, $\Phi_3(S_3^+)$ is also a line, and this can also be written as $\Phi_3(S_3^+ \cap D)$. This implies that $\Phi_3(D)$ contains the line $\Phi_3(S_3^+)$ too. If $\Phi_3(S_3^+) = \Phi_3(S_3^-)$, the image $\Phi(S_3^- \cup S_3^+)$ would be a 1-dimensional linear subspace of the plane $\pi_3^{-1}(\lambda_3)$, which cannot happen. Thus the two lines $\Phi_3(S_3^+)$ and $\Phi_3(S_3^-)$ are distinct. Hence $\Phi_3(D)$ contains two lines. Therefore $\Phi_3(D) = \mathbb{CP}^2$, and we finally obtain $(\text{Bs } |\mathcal{L}_3|) \cap D = \emptyset$. Thus we conclude $\text{Bs } |\mathcal{L}_3| = \emptyset$, as claimed. \square

For restrictions of Φ_3 to fibers of the morphism $f_3 : Z_3 \rightarrow \mathbb{CP}^1$, we have the following.

Lemma 4.3. (i) If $S = f_3^{-1}(\lambda)$ is a smooth fiber of f_3 , the restriction $\Phi_3|_S$ is of degree two over the plane $\pi^{-1}(\lambda)$. (ii) On two reducible fibers $S_1^+ \cup S_1^-$ and $S_2^+ \cup S_2^-$, Φ_3 is birational over the plane $\pi^{-1}(\lambda_1)$ and $\pi^{-1}(\lambda_2)$ respectively on each irreducible component. Further, the image of the twistor line $L_i = S_i^+ \cap S_i^-$ is a conic in the plane. (iii) Φ_3 contracts each of S_3^+ and S_3^- to a line in the plane. Further, these two lines are distinct.

In terms of the original map $\Phi : Z \rightarrow \mathbb{CP}^4$, the above (iii) means that the anticanonical map contracts S_3^+ and S_3^- to lines. This is a major difference between the case of type I, where the anticanonical map does not contract any divisor [4, Proposition 3.6].

Proof of Lemma 4.3. (i) is obvious because on such S the restriction $\Phi_3|_S$ can be identified with the bi-anticanonical map of S , which is degree two over a plane by Proposition 2.3 (iii). For (ii) in the case $i = 1$, on Z_3 we temporarily put $C_2 = S_1^+ \cap E_2, C_3 = S_1^+ \cap E_3$ and $\overline{C}_1 = S_1^+ \cap \overline{E}_1$. We also put $\Delta_1 = S_1^+ \cap E_1$ (see lower left in Figure 2). Then by the isomorphism (4.2) we have $\mathcal{L}_3|_{S_1^+} \simeq \mathcal{O}_{S_1^+}(\Delta_1 + C_2 + 2C_3 + \overline{C}_1)$. By standard computations it is possible to show that:

$$\begin{aligned} h^0(S_1^+, \mathcal{O}(\Delta_1 + C_2 + 2C_3 + \overline{C}_1)) &= 3, \\ (\Delta_1 + C_2 + 2C_3 + \overline{C}_1)^2 &= 1 \text{ on } S_1^+, \\ \text{Bs } |\Delta_1 + C_2 + 2C_3 + \overline{C}_1| &= \emptyset, \end{aligned}$$

and also the induced morphism $S_1^+ \rightarrow \mathbb{CP}^2$ is birational. The restriction $\Phi_3|_{S_1^+}$ is nothing but the rational map associated to the image of the restriction map $H^0(Z_3, \mathcal{L}_3) \rightarrow H^0(S_1^+, \mathcal{O}_{S_1^+}(\Delta_1 + C_2 + 2C_3 + \overline{C}_1))$. The last image cannot be 0 or 1-dimensional since $\text{Bs } |\mathcal{L}_3| \neq \emptyset$. Also, it cannot be 2-dimensional since in that case $|\mathcal{L}_3|_{S_1^+}|$ would have a base point because $(\mathcal{L}_3|_{S_1^+})^2 = 1$ as above. Thus we conclude that $\Phi_3|_{S_1^+}$ is birational. By similar computations for which we omit, we can also show that $\Phi_3|_{S_2^+}$ is birational over the plane $\pi^{-1}(\lambda_2)$.

(iii) is already shown in the final part of the proof of Proposition 4.2. \square

The morphism Φ_3 maps the exceptional divisors of the blowups as follows.

Proposition 4.4. Let $\Phi_3 : Z_3 \rightarrow \mathbb{CP}^4$ be the morphism associated to $|\mathcal{L}_3|$ as before. Then (i) by Φ_3 , the divisors E_1 and \overline{E}_3 are mapped to one and the same point on the singular line of the scroll Y . The same is true for \overline{E}_1 and E_3 . (ii) by Φ_3 , E_2 and \overline{E}_2 are mapped onto the singular line of Y . (iii) by Φ_3 , D and \overline{D} are mapped birationally to the plane $\pi^{-1}(\lambda_3)$.

Proof. We first show (ii). As is remarked in the proof of Proposition 4.2, the restriction $|\mathcal{L}_3|_{E_2}|$ is a pencil without a base point. Therefore by the same reason for S_3^+ in Lemma 4.3, E_2 is mapped to a line by Φ_3 . We show this line must be the singular line of the scroll. For any non-singular fiber S of f_3 , $\Phi_3|_S$ is naturally identified with the bi-anticanonical map of S , and $S \cap E_2$ is exactly the component C_2 , which is mapped to a line by Proposition 2.3 (v). Since this line is exactly $\Phi(C_2)$, this is independent of the choice of S . Therefore the line must be the intersection of fiber planes of the projection $\pi : \mathbb{CP}^4 \rightarrow \mathbb{CP}^2$. Thus the $\Phi_3(E_2)$ must be the singular line of Y . Hence since the singular line is real, \overline{E}_2 is also mapped to the same line, and we get (ii).

Since \mathcal{L}_3 is trivial on E_1 and \overline{E}_3 , each of these are mapped to a point by Φ_3 . Since $E_1 \cap \overline{E}_3 \neq \emptyset$, these points must coincide. Further, as $E_1 \cap E_2 \neq \emptyset$ the last point must belong to the singular line of the scroll. By real structure we obtain the same conclusion for \overline{E}_1 and E_3 , and we obtain (i). Finally (iii) is already shown in the course of the proof of Proposition 4.2. \square

4.2. The case of type III. The elimination of the base locus of the anticanonical map in the case of type III can be done along the same line as in the case of type II (though more complicated). So the description below is partially sketchy.

Let Z be a twistor space on $4\mathbb{CP}^2$ with degree two anticanonical map, which is of type III. Let $\mu_1 : Z_1 \rightarrow Z$ be the blowup at C , and E_i and \overline{E}_i ($1 \leq i \leq 4$ this time) the exceptional divisors over C_i and \overline{C}_i respectively. Then again Z_1 admits a morphism $f_1 : Z_1 \rightarrow \mathbb{CP}^1$ induced by the pencil

$$\left| \mu_1^* F - \sum_{1 \leq i \leq 4} (E_i + \overline{E}_i) \right|.$$

This fibration f_1 has exactly four reducible fibers and they can be described as in the first column of Figure 3 in a neighborhood of the fibers. Let λ_i , $1 \leq i \leq 4$, be the points on Λ corresponding to the reducible fiber $S_i^+ \cup S_i^-$. As in the case of type II, Z_1 has exactly two ordinary nodes on each reducible fiber (again indicated as dotted points). For each of these eight nodes of Z_1 we take a small resolution as displayed in the figure. Let $\mu_2 : Z_2 \rightarrow Z_1 \xrightarrow{\mu_1} Z$ be the composition, and put

$$\mathcal{L}_2 := \mu_2^*(2F) - (E_1 + E_2 + E_3 + \overline{E}_1 + \overline{E}_2 + \overline{E}_3).$$

(See Proposition 2.4 (i).) Let $f_2 : Z_2 \rightarrow Z_1 \xrightarrow{f_1} \mathbb{CP}^1$ be the composition. We have a natural isomorphism $H^0(2F) \simeq H^0(\mathcal{L}_2)$ and also, similarly to (4.2) in the case of type II, an isomorphism

$$(4.4) \quad \mathcal{L}_2 \simeq f_2^* \mathcal{O}_{\mathbb{CP}^1}(2) + E_1 + E_2 + E_3 + 2E_4 + \overline{E}_1 + \overline{E}_2 + \overline{E}_3 + 2\overline{E}_4.$$

Then analogously to Lemma 4.1 we have the following

Lemma 4.5. *On Z_2 , we have the following: (i) the system $|\mathcal{L}_2|$ does not have a fixed component, (ii) the following four smooth rational curves*

$$(4.5) \quad S_4^+ \cap \overline{E}_1, \quad S_4^+ \cap \overline{E}_2, \quad S_4^- \cap E_1, \quad S_4^- \cap E_2$$

are base curves of $|\mathcal{L}_2|$. (Note that the first two curves intersect and the same for the last two curves; see Figure 3.)

Proof. (i) is completely analogous to the proof of Lemma 4.1 (i). (We use Proposition 2.4 (ii) instead of Proposition 2.3 (ii).) For (ii), from (4.4) and Figure 3, the intersection numbers

of \mathcal{L}_2 with the four curves (4.5) are respectively computed to be $-1, 0, -1, 0$. These imply the claim (ii). \square

Let $\mu_3 : Z_3 \rightarrow Z_2$ be the blowup at the base curves (4.5) and D_1 and D_2 the exceptional divisors over $S_4^- \cap E_1$ and $S_4^- \cap E_2$ respectively. A careful computation shows

$$(4.6) \quad D_1 \simeq \mathbb{CP}^1 \times \mathbb{CP}^1, \quad D_2 \simeq \mathbb{P}(\mathcal{O}(1) \oplus \mathcal{O}).$$

Z_3 has ordinary nodes over the two singular points of the center of μ_3 . Let $Z_4 \rightarrow Z_3$ be the small resolution of these two nodes as displayed in Figure 3, and $\mu_4 : Z_4 \rightarrow Z_3 \rightarrow Z_2$ the composition. Then by the last small resolution D_1 is blownup at a point, while D_2 remains unchanged (see Figure 3). We again use the same letters to denote the strict transforms of these divisors into Z_4 , and put

$$\mathcal{L}_4 := \mu_4^* \mathcal{L}_2 - (D_1 + D_2 + \overline{D}_1 + \overline{D}_2).$$

Then similarly to Proposition 4.2, we have

Proposition 4.6. *The system $|\mathcal{L}_4|$ on Z_4 is base point free.*

Proof. By Lemma 4.5 (i) the divisors E_i and \overline{E}_i are not a fixed component of $|\mathcal{L}_4|$ for any $1 \leq i \leq 4$. Further by using the relation (4.4) and chasing Figure 3 it is possible to show that the line bundle \mathcal{L}_4 is trivial on E_1, E_2 and E_4 . These mean that $|\mathcal{L}_4|$ does not have a base point on E_1, E_2 and E_4 . Hence, with the aid of Proposition 2.5 (ii), we have

$$\text{Bs } |\mathcal{L}_4| \subset E_3 \cup D_1 \cup D_2 \cup \overline{E}_3 \cup \overline{D}_1 \cup \overline{D}_2.$$

Moreover, the system $|\mathcal{L}_4|_{E_3}$ satisfies $\dim |\mathcal{L}_4|_{E_3}| = 1$ and the intersection number with a fiber class of the natural projection $E_3 \rightarrow \Lambda$ is one. Therefore, since $|\mathcal{L}_4|$ does not have a base point on a smooth fiber of the composition $Z_4 \rightarrow Z_1 \rightarrow \Lambda$, we obtain that $|\mathcal{L}_4|$ does not have a base point on E_3 too.

Let $\Phi_4 : Z_4 \rightarrow Y \subset \mathbb{CP}^4$ be the map associated to $|\mathcal{L}_4|$. It remains to see that there is no base point on D_1 nor D_2 . By (4.6), D_2 is biholomorphic to one point blowup of \mathbb{CP}^2 . Further, again from Figure 3 and (4.4), we can deduce that $\mathcal{L}_4|_{D_2}$ is isomorphic to the pullback of $\mathcal{O}(1)$ by a blowdown $D_2 \rightarrow \mathbb{CP}^2$. Also it is possible to deduce that the restrictions of \mathcal{L}_4 to the divisors S_4^- and D_1 are respectively isomorphic to the pullback of $\mathcal{O}_{\mathbb{CP}^1}(1)$ by surjective morphisms $S_4^- \rightarrow \mathbb{CP}^1$ and $D_1 \rightarrow \mathbb{CP}^1$, the curves $S_4^- \cap D_2$ and $S_4^- \cap \overline{D}_1$ are sections of the morphism $S_4^- \rightarrow \mathbb{CP}^1$, and that both $D_1 \cap S_4^+$ and $D_1 \cap D_2$ are sections of the morphism $D_1 \rightarrow \mathbb{CP}^1$. (In Figure 3 fibers of these morphisms are indicated by dotted curves.) From these we obtain that the restrictions of Φ_4 to S_4^- and D_1 are exactly the above morphisms to \mathbb{CP}^1 , and that the image $\Phi_4(S_4^-)$ and $\Phi_4(D_1)$ are lines in the plane $\pi^{-1}(\lambda_4)$. In particular we have $(\text{Bs } |\mathcal{L}_4|) \cap D_1 = \emptyset$. We also have $\Phi_4(S_4^-) = \Phi_4(S_4^- \cap \overline{D}_1)$ and $\Phi_4(D_1) = \Phi_4(D_1 \cap D_2) = \Phi_4(D_1 \cap S_4^+)$. Hence by the real structure we have $\Phi_4(D_1 \cap D_2) = \Phi_4(S_4^+)$. Therefore the above two lines in $\pi^{-1}(\lambda_4)$ can be rewritten as $\Phi_4(S_4^-)$ and $\Phi_4(S_4^+)$, which means that these are distinct. Thus the image $\Phi_4(D_2)$ contains two distinct lines and hence $\Phi_4|_{D_2} : D_2 \rightarrow \Phi_4(D_2)$ is just the blowdown (of the (-1) -curve $D_2 \cap E_2$). In particular, $(\text{Bs } |\mathcal{L}_4|) \cap D_2 = \emptyset$. Hence we obtain $\text{Bs } |\mathcal{L}_4| = \emptyset$. \square

By the proposition, the map $\Phi_4 : Z_4 \rightarrow Y \subset \mathbb{CP}^4$ is a morphism. For the images of divisors on Z_4 by Φ_4 , we have the following

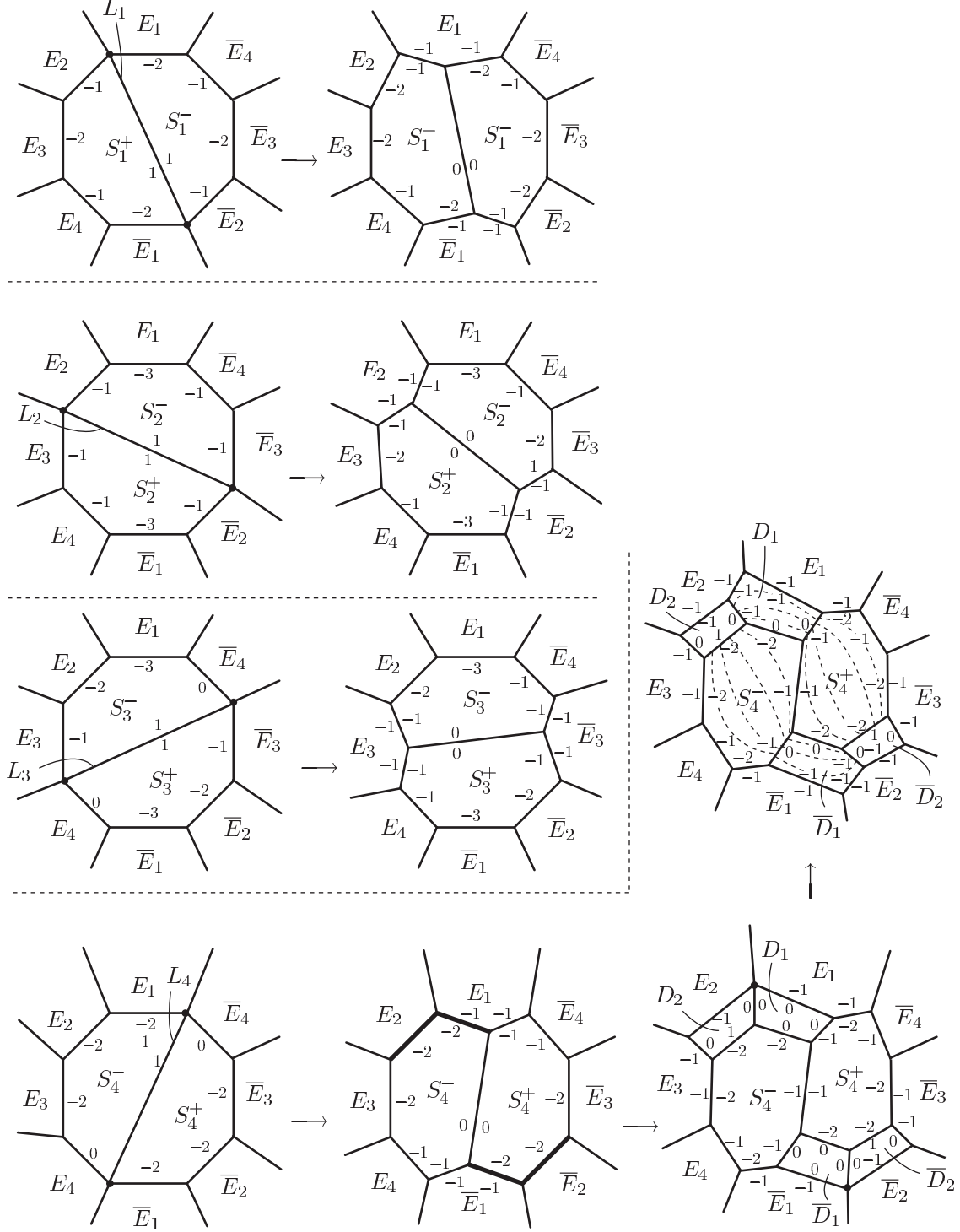


FIGURE 3. The blowups for the case of type III.

Proposition 4.7. *(i) For $i = 1, 2, 3$, the restrictions of Φ_4 to the divisors S_i^\pm are birational over the plane $\pi^{-1}(\lambda_i)$, (ii) Φ_4 contracts S_4^+ and S_4^- to lines in $\pi^{-1}(\lambda_4)$ which are mutually distinct, (iii) for $i = 1, 2, 4$, Φ_4 contracts the divisors E_i and \overline{E}_i to points on the singular line l of the scroll, (iv) Φ_4 maps E_3 and \overline{E}_3 to the line l , (v) $\Phi_4(D_1) = \Phi_4(S_4^+)$ and $\Phi_4(\overline{D}_1) = \Phi_4(S_4^-)$, (vi) the restrictions of Φ_4 to D_2 and \overline{D}_2 are birational over the plane $\pi^{-1}(\lambda_4)$.*

Proof. The claims (ii), (iii), (v) and (vi) are already proved in the proof of Proposition 4.6. (i) and (iv) can be shown in a similar way to Lemma 4.3 (i) and Proposition 4.4 (ii) respectively. \square

4.3. The branch divisor of the anticanonical map. Let Z be a twistor space on $4\mathbb{CP}^2$, which is of type II or type III, and $\Phi : Z \rightarrow Y \subset \mathbb{CP}^4$ the anticanonical map as before. We write $\nu : \tilde{Y} \rightarrow Y$ for the blowup of Y along the singular line l . Since Λ is a conic, \tilde{Y} is biholomorphic to the total space of the \mathbb{CP}^2 -bundle $\mathbb{P}(\mathcal{O}(2)^{\oplus 2} \oplus \mathcal{O}) \rightarrow \mathbb{CP}^1 \simeq \Lambda$. We say that a morphism to Y can be lifted to \tilde{Y} if it factors through the resolution $\tilde{Y} \rightarrow Y$.

Lemma 4.8. *(i) If Z is of type II, the morphism $\Phi_3 : Z_3 \rightarrow Y$ obtained in Section 4.1 can be lifted to a morphism $\tilde{\Phi}_3 : Z_3 \rightarrow \tilde{Y}$. (ii) If Z is of type III, the morphism $\Phi_4 : Z_4 \rightarrow Y$ obtained in Section 4.2 can be lifted to a morphism $\tilde{\Phi}_4 : Z_4 \rightarrow \tilde{Y}$.*

Proof. Since this can be shown as in the proof of the corresponding statement [4, Proposition 3.4], we just give a sketch of the proof. The indeterminacy locus of the rational map $f : Z \rightarrow \mathbb{CP}^1$ associated to the pencil $|F|$ was eliminated by just blowing-up the cycle C , and we always have a morphism to \mathbb{CP}^1 from the series of blowup spaces. Via the natural identification between fibers of $\tilde{Y} \rightarrow \Lambda$ and planes in the scroll Y , this gives the desired lifts of the morphisms $\Phi_3 : Z_3 \rightarrow Y$ and $\Phi_4 : Z_4 \rightarrow Y$. \square

As the original anticanonical map $\Phi : Z \rightarrow Y$ is (rational but) of degree two, the lifts are also of degree two. For the branch divisor of these lifts, we have

Proposition 4.9. *Let $\tilde{\Phi}_3 : Z_3 \rightarrow \tilde{Y}$ and $\tilde{\Phi}_4 : Z_4 \rightarrow \tilde{Y}$ be the lifts as in Lemma 4.8. Then the branch divisor of these degree-two morphisms are pull-back by $\nu : \tilde{Y} \rightarrow Y$ of a divisor belonging to the system $|\mathcal{O}_Y(4)|$, where $\mathcal{O}_Y(4) := \mathcal{O}_{\mathbb{CP}^4}(4)|_Y$.*

Because this can be proved in a similar way to the case of type I ([4, Proposition 3.4]), we just mention that it is enough to use Propositions 2.3 and 2.4 instead of [4, Propositions 2.2 and 2.3].

By Proposition 4.9, the branch divisors of the morphisms $\Phi_3 : Z_3 \rightarrow Y$ and $\Phi_4 : Z_4 \rightarrow Y$ are cuts of Y by quartic hypersurfaces. Needless to say, this hyperquartic is the most significant data for determining the twistor spaces. In the next section we examine defining equation of these hyperquartics.

5. DEFINING EQUATIONS OF THE BRANCH DIVISORS

As in the previous section let Z be a Moishezon twistor space on $4\mathbb{CP}^2$ whose anticanonical map is degree two which is of type II or type III, and $\Phi_3 : Z_3 \rightarrow Y$ or $\Phi_4 : Z_4 \rightarrow Y$ respectively be the degree two morphisms which are obtained as a consequence of the elimination of the original anticanonical map $\Phi : Z \rightarrow Y$, given in the last section. We denote by $B (\subset Y)$ the branch divisor of the morphisms. By Proposition 4.9 we know that B is of the form $Y \cap \mathcal{B}$, where \mathcal{B} is a quartic hypersurface in \mathbb{CP}^4 .

5.1. Double curves on the branch divisor. Let H be a hyperplane in \mathbb{CP}^4 . The intersection $Y \cap H$ is either a plane or a cone over the conic Λ . We say that a curve \mathcal{C} on the branch divisor B is a *double curve with respect to H* if the hyperplane H touches B along the curve \mathcal{C} ; or more precisely if $B \cap H$ is a non-reduced curve on a reduced surface $Y \cap H$ (i.e. a plane or a cone). In this subsection we find five double curves on B .

First we consider the case of type II. For $i = 1, 2$ let $L_i = S_i^+ \cap S_i^-$ be the intersection twistor line. Then by Lemma 4.3 the image $\Phi_3(L_i)$, which clearly coincides with $\Phi(L_i)$, is a plane conic. We put $\mathcal{C}_i := \Phi(L_i)$. Then as Φ is birational on each of the components S_1^\pm and S_2^\pm by the lemma, \mathcal{C}_1 and \mathcal{C}_2 are double curves with respect to hyperplanes containing the planes $\pi^{-1}(\lambda_1)$ and $\pi^{-1}(\lambda_2)$ respectively. We call these two curves as *double conics*.

On the other hand the situation is very different for S_3^+ and S_3^- by Lemma 4.3 (iii) in that these are contracted to lines by Φ . We define

$$\mathcal{C}_3 := \Phi(S_3^+ \cup S_3^-).$$

By the lemma this is a union of two distinct lines in the plane $\pi^{-1}(\lambda_3)$. From the definition the curve \mathcal{C}_3 must also be a double curve with respect to hyperplanes containing $\pi^{-1}(\lambda_3)$. We call \mathcal{C}_3 a *splitting double conic*. This kind of a double curve did not appear in the case of type I and will be significant in obtaining a strong constraint for defining equation of the branch divisor. These three double conics $\mathcal{C}_1, \mathcal{C}_2$ and \mathcal{C}_3 are placed in \mathbb{CP}^4 in a way that the intersection $\mathcal{C}_i \cap l$ (where l is the singular line of the scroll as before) consists of two points which are independent of i . More invariantly, recalling that the bi-anticanonical map ϕ on S contracts the connected curves $\overline{C}_3 \cup C_1$ and $C_3 \cup \overline{C}_1$ (Proposition 2.3 (iv)), these two points on l are the images $\Phi(\overline{C}_3 \cup C_1)$ and $\Phi(C_3 \cup \overline{C}_1)$ respectively.

Next recalling that the anticanonical system $|2F|$ has two distinguished members $X_1 + \overline{X}_1$ and $X_2 + \overline{X}_2$ by Proposition 3.1 (i) we define H_i ($i = 4, 5$) to be the unique hyperplane in \mathbb{CP}^4 which satisfies $\Phi^{-1}(H_i) = X_{i-3} + \overline{X}_{i-3}$. We now show that X_1 and X_2 are not contracted to a curve or a point by Φ . For this as in the proof of Proposition 3.1 for a generic member $S \in |F|$ the intersection $S \cap X$ contains a curve which does not contain any component of the cycle C . By using $K_S^2 = 0$, it is easy to show that the curves C_1, C_3, \overline{C}_1 and \overline{C}_3 are all curves that are contracted to a point by the bi-anticanonical map ϕ of S . Hence recalling $\Phi|_S = \phi$, the above curve $X \cap S$ is not contracted to a point by Φ . Because S is generic, this implies that X itself is also not contracted to a curve by Φ , as claimed. Therefore $\Phi|_{X_i}$ is birational, and $\Phi(X_i \cap \overline{X}_i)$ ($i = 1, 2$) must be a double curve with respect to H_{i+3} . We write \mathcal{C}_4 and \mathcal{C}_5 for these double curves. As Y is quadratic and $B \in |\mathcal{O}_Y(4)|$, these curves must be of degree four in \mathbb{CP}^4 . We call these *double quartic curves*. These two curves intersect at four points on the plane $H_4 \cap H_5$. So for the case of type II we have obtained three double conics (one of which is a splitting one) and two double quartic curves.

For the case of type III, in a similar way, the three curves $\mathcal{C}_i := \Phi(L_i)$ (where $L_i = S_i^+ \cap S_i^-$ and $1 \leq i \leq 3$ this time) are double conics with respect to hyperplanes containing the plane $\pi^{-1}(\lambda_i)$. Also if we put $\mathcal{C}_4 := \Phi(S_4^+ \cup S_4^-)$, this becomes a splitting double conic in the above sense by Proposition 4.7 (ii). These four double conics intersect the singular line l at two points which are independent of the choice of the conic. These two points are nothing but the image under Φ of the connected curves $\overline{C}_4 \cup C_1 \cup C_2$ and $C_4 \cup \overline{C}_1 \cup \overline{C}_2$. Further recalling Proposition 3.1 (ii) and letting H_5 be the hyperplane satisfying $\Phi^{-1}(H_5) = X + \overline{X}$, $\mathcal{C}_5 := \Phi(X \cap \overline{X})$ becomes a double quartic curve with respect to H_5 by the same reason for

\mathcal{C}_4 and \mathcal{C}_5 in the case of type II. Thus for the case of type III we have obtained four double conics (one of which is a splitting one) and one double quartic curve.

We emphasize that for both of types II and III double conics are lying on fiber planes of the scroll $Y \rightarrow \Lambda$, whereas double quartic curves are lying on cones which are hyperplane sections of the scroll. This difference will be significant when we determine defining equation of the branch divisor of the anticanonical map of the twistor spaces.

Remark 5.1. We now display the number of double curves, including the cases of type I and type IV. For the case of type I, these are proved in [4, Section 4.1]. The case of type IV can be proved in a similar way. (But note that in this case we did not give a full elimination of the base locus in [3].)

	type I	type II	type III	type IV
double conics (splitting one)	2 (0)	3 (1)	4 (1)	5 (1)
double quartic curves	3	2	1	0
total number	5	5	5	5

Note that the number of the double conics is always equal to the number of the reducible members of the pencil $|F|$.

5.2. Quadratic hypersurfaces containing double curves. Next we investigate hyperquadrics in \mathbb{CP}^4 containing all these five double curves.

Proposition 5.2. *As before let Z be of type II or type III, $B \subset Y$ the branch divisor of the anticanonical map, and \mathcal{C}_i , $1 \leq i \leq 5$, the double curves on B . Then there exists a hyperquadric in \mathbb{CP}^4 which contains all these double curves and which is different from the scroll Y .*

Proof. For the case of type II, we consider quadratic hypersurfaces in \mathbb{CP}^4 which go through all the following 12 points:

- (a) $\mathcal{C}_1 \cap \mathcal{C}_2$ (consisting of 2 points),
- (b) $\mathcal{C}_4 \cap \mathcal{C}_5$ (consisting of 4 points),
- (c) $\mathcal{C}_1 \cap \mathcal{C}_4$, and one of the two points $\mathcal{C}_1 \cap \mathcal{C}_5$ (consisting of 3 points),
- (d) $\mathcal{C}_2 \cap \mathcal{C}_4$ (consisting of 2 points),
- (e) one of the 2 points $\mathcal{C}_2 \cap \mathcal{C}_5$ (consisting of 1 point).

We show that if a hyperquadric Q goes through these 12 points, then Q automatically contains all the five double curves. In fact, from (a)–(c), Q contains 5 points on \mathcal{C}_1 , and hence $\mathcal{C}_1 \subset Q$. If Q further goes through the two points in (d), Q passes through 8 points on \mathcal{C}_4 , which means $\mathcal{C}_4 \subset Q$. Furthermore from the final point (e), Q goes through 5 points on \mathcal{C}_2 , which means $\mathcal{C}_2 \subset Q$. This implies that Q passes through 8 points on \mathcal{C}_5 , and therefore $\mathcal{C}_5 \subset Q$. These mean that Q contains 6 points on \mathcal{C}_3 , meaning $\mathcal{C}_3 \subset Q$. Thus the hyperquadric Q contains all the five double curves. Since $h^0(\mathcal{O}_{\mathbb{CP}^4}(2)) = 15$, these hyperquadrics form a 2-dimensional subsystem. Any one of these Q -s which is different from Y gives the required quadratic hypersurface.

The case of type III can be shown in a similar way. We omit the detail. \square

5.3. Defining equation of the branch divisor. We are ready for determining defining equation of the branch divisor of the anticanonical map. As before let $\pi : \mathbb{CP}^4 \rightarrow \mathbb{CP}^2$ be a linear projection, Λ a conic in the target plane, and $Y \subset \mathbb{CP}^4$ the scroll over Λ . We know Y is the image of the anticanonical map of the twistor spaces. We fix any homogeneous

coordinates (z_0, z_1, z_2) on the above plane such that the conic Λ is defined by the equation $z_0^2 = z_1 z_2$.

Theorem 5.3. *Let Z be a Moishezon twistor space on $4\mathbb{CP}^2$ whose anticanonical map is (rationally) of degree two over the image, and $\Phi : Z \rightarrow Y \subset \mathbb{CP}^4$ the anticanonical map. Then the branch divisor of Φ is an intersection of the scroll Y with a quartic hypersurface defined by the following equation:*

(i) *If Z is of type II, the equation is of the form*

$$(5.1) \quad z_0 z_1 z_3 z_4 = Q(z_0, z_1, z_2, z_3, z_4)^2,$$

where Q is a quadratic polynomial such that the discriminant of the quadratic form $Q(0, 0, z_2, z_3, z_4)$ is zero.

(ii) *If Z is of type III, the equation of the hyperquartic is of the form*

$$(5.2) \quad z_0 z_1 z_4 (z_0 - \lambda z_1) = Q(z_0, z_1, z_2, z_3, z_4)^2,$$

where λ is a real number satisfying $\lambda \neq 0, 1$, and Q is a quadratic polynomial such that the discriminant of the quadratic form $Q(0, 0, z_2, z_3, z_4)$ is zero.

Remark 5.4. Before proceeding to the proof, because of the coherency of the four types of the twistor spaces, we here write the equations of the branch divisors in the cases of type I and type IV. For the case of type I, under the above normalization for the equation of the conic Λ , the equation of the hyperquartic is of the form

$$(5.3) \quad z_0 z_3 z_4 f(z_0, z_1, z_2, z_3, z_4) = Q(z_0, z_1, z_2, z_3, z_4)^2,$$

where f and Q are linear and quadratic forms respectively. (No constraint for the discriminant of Q for this case.) For the case of type IV, under the same normalization, the equation is of the form

$$(5.4) \quad z_0 z_1 (z_0 - \lambda_1 z_1)(z_0 - \lambda_2 z_1) = Q(z_0, z_1, z_2, z_3, z_4)^2,$$

where λ_1 and λ_2 are distinct real numbers satisfying $\lambda_1, \lambda_2 \notin \{0, 1\}$. Further the discriminant of the quadratic form $Q(0, 0, z_2, z_3, z_4)$ vanishes.

Although these might look complicated, a general principle is simple: regardless of the types, the equation of the hyperquartic is always of the form

$$(5.5) \quad (\text{product of four linear polynomials}) = Q(z_0, z_1, z_2, z_3, z_4)^2,$$

and according to each of the degenerations $\text{I} \rightarrow \text{II} \rightarrow \text{III} \rightarrow \text{IV}$, one of the linear polynomial degenerates from those with 5 variables z_0, z_1, z_2, z_3, z_4 to those with 3 variables z_0, z_1, z_2 ; geometrically the hyperplanes (in \mathbb{CP}^4) defined by the linear polynomials degenerate from general ones to those which contain the singular line l of the scroll Y . So the ‘absolute value’ of the type coincides with the number of the linear forms in the left-hand-side of (5.5) which belong to $\mathbb{C}[z_0, z_1, z_2]$.

Thus there is a strong similarity with the case of $3\mathbb{CP}^2$ obtained by Kreussler-Kurke [7, p. 49, 50] (see also Poon [9]).

Proof of Theorem 5.3. For an algebraic subset $X \subset \mathbb{CP}^n$, we denote by $I_X \subset \mathbb{C}[z_0, \dots, z_n]$ the homogeneous ideal of X . By Proposition 4.9 there exists a hyperquartic \mathcal{B} in \mathbb{CP}^4 such that the branch divisor B of the anticanonical map is given as $Y \cap \mathcal{B}$. Let $F = F(z_0, \dots, z_4)$ be a defining quartic polynomial of \mathcal{B} , where z_0, \dots, z_4 are homogeneous coordinates on \mathbb{CP}^4 , chosen in such a way that the projection $\pi : \mathbb{CP}^4 \rightarrow \mathbb{CP}^2$ is given by $(z_0, \dots, z_4) \mapsto$

(z_0, z_1, z_2) . In particular we have $l = \{z_0 = z_1 = z_2\}$. Obviously F is defined only up to an ideal $I_Y \subset \mathbb{C}[z_0, \dots, z_4]$. Let Q be a defining polynomial of the hyperquadrics in \mathbb{CP}^4 containing all the double conics, whose existence was proved in Proposition 5.2. Let $\mathbb{P}_i \subset Y$ be the plane $\pi^{-1}(\lambda_i)$, where $1 \leq i \leq 3$ in the case of type II and $1 \leq i \leq 4$ in the case of type III. Then as $(F|_{\mathbb{P}_i}) = 2\mathcal{C}_i = (Q^2|_{\mathbb{P}_i})$ as divisors on the plane \mathbb{P}_i , there exists a constant c_i such that $F - c_i Q^2 \in I_{\mathbb{P}_i} \subset \mathbb{C}[z_0, \dots, z_4]$. If $c_i \neq c_j$ for some $i \neq j$, we obtain $Q^2 \in I_{\mathbb{P}_i} + I_{\mathbb{P}_j}$. Further the last ideal is readily seen to be equal to $I_{\mathbb{P}_i \cap \mathbb{P}_j}$, and therefore equals to $I_l = (z_0, z_1, z_2) \subset \mathbb{C}[z_0, \dots, z_4]$. Hence $Q \in (z_0, z_1, z_2)$. But this means that the divisor $(Q|_{\mathbb{P}_i})$ contains l , which contradicts the structure of the double conics (including the splitting one) obtained in Section 5.1. Therefore $c_i = c_j$ for any double conics \mathcal{C}_i and \mathcal{C}_j .

Next for the double quartic curve \mathcal{C}_k , so that $k = 4, 5$ in the case of type II and $k = 5$ for the case of type III, since $(F|_{H_k \cap Y}) = 2\mathcal{C}_k = (Q^2|_{H_k \cap Y})$, there exists a constant c_k such that $F - c_k Q^2 \in I_{H_k \cap Y} = (z_k) + I_Y$. So taking a difference with $F - c_1 Q^2 \in I_{\mathbb{P}_1}$, we obtain that $(c_1 - c_k)Q^2 \in (z_k) + I_Y + I_{\mathbb{P}_1}$. But since $\mathbb{P}_1 \subset Y$, we have $I_{\mathbb{P}_1} \supset I_Y$, and therefore $(c_1 - c_k)Q^2 \in (z_k) + I_{\mathbb{P}_1}$. Hence if $c_1 \neq c_k$ we have $Q^2 \in (z_k) + I_{\mathbb{P}_1}$, which means $Q^2|_{\mathbb{P}_1} \in (z_k|_{\mathbb{P}_1})$. Since $k > 2$, this means that the divisor $(Q^2)|_{\mathbb{P}_1}$ contains a line (z_k) on the plane \mathbb{P}_1 as an irreducible component, which again contradicts the irreducibility of \mathcal{C}_1 . Therefore we have $c_1 = c_k$ for any double quartic curve \mathcal{C}_k . By rescaling we can suppose $c_i = 1$ for any $1 \leq i \leq 5$. Thus we have

$$(5.6) \quad F - Q^2 \in I_{\mathbb{P}_1} \cap I_{\mathbb{P}_2} \cap I_{\mathbb{P}_3} \cap ((z_3) + I_Y) \cap ((z_4) + I_Y) \quad \text{for the case of type II,}$$

$$(5.7) \quad F - Q^2 \in I_{\mathbb{P}_1} \cap I_{\mathbb{P}_2} \cap I_{\mathbb{P}_3} \cap I_{\mathbb{P}_4} \cap ((z_4) + I_Y) \quad \text{for the case of type III.}$$

For the case of type III, by using the last ideal, we can write $F - Q^2 = z_4 f + g$, where f is a cubic polynomial and g is a quartic polynomial in I_Y . This readily means $z_4 f \in I_{\mathbb{P}_i}$ for any double conic \mathcal{C}_i (namely for $i = 1, 2, 3, 4$). For $i = 1, 2, 3$, let l_i be the line going through the two points λ_i and λ_4 , and f_i a defining linear polynomial of l_i . Then we have

$$(5.8) \quad Y \cap \pi^{-1}(l_i) = \mathbb{P}_i \cup \mathbb{P}_4, \quad 1 \leq i \leq 3,$$

and therefore

$$I_{\mathbb{P}_i} \cap I_{\mathbb{P}_4} = I_{\mathbb{P}_i \cup \mathbb{P}_4} = I_{Y \cap \pi^{-1}(l_i)} = I_Y + I_{\pi^{-1}(l_i)}.$$

Hence from $z_4 f \in I_{\mathbb{P}_i} \cap I_{\mathbb{P}_4}$ we can write $z_4 f = yg + f_i h$ where y is a defining quadratic polynomial of the scroll Y . If we write $g = z_4 g_1 + g_2$ and $h = z_4 h_1 + h_2$ in a way that g_2 and h_2 do not involve z_4 , then we compute $yg + f_i h = z_4(yg_1 + f_i h_1) + (yg_2 + f_i h_2)$. As this equals $z_4 f$ and $yg_2 + f_i h_2$ does not involve z_4 , we obtain $yg_2 + f_i h_2 = 0$. Hence, since y and f_i are mutually prime from irreducibility of the conic Λ , we can write $h_2 = y h_3$ by some quadratic polynomial h_3 . Thus we obtain

$$z_4 f = z_4 f_i h_1 + y(g + f_i h_3).$$

Repeating a similar argument we can pull out the linear polynomials f_1, f_2 and f_3 one by one, and it follows that $z_4 f$ can be written of the form $z_4 f_1 f_2 f_3 + \eta$, where $\eta \in I_Y$. Therefore we obtain

$$(5.9) \quad F - Q^2 = z_4 f_1 f_2 f_3 + (g + \eta), \quad g + \eta \in I_Y.$$

By usual $\text{PGL}(3, \mathbb{C})$ -action we can normalize the homogeneous coordinates (z_0, z_1, z_2) in such a way that

$$\lambda_4 = (0, 0, 1), \quad \lambda_1 = (0, 1, 0), \quad \lambda_2 = (1, 0, 0)$$

hold. Then we may suppose $f_1 = z_0$ and $f_2 = z_1$. Moreover, as $l_3 \ni \lambda_4$ and all λ_i -s are real and mutually distinct, f_3 must be of the form $z_0 - \lambda z_1$, where $\lambda \in \mathbb{R} \setminus \{0, 1\}$. Thus disposing $g + \eta$, (5.9) means that a defining equation of the hyperquartic \mathcal{B} can be taken in the form (5.2) in Theorem 5.3.

For the case III, we still remain to show the claim about discriminant of Q . As $\lambda_4 = (0, 0, 1)$ from the above choice of the coordinates, the image $\Phi(S_4^+)$ and $\Phi(S_4^-)$ and over the plane $\{z_0 = z_1 = 0\}$. Further as in Proposition 4.7 (ii), these images are mutually distinct lines. Hence $Q(0, 0, z_2, z_3, z_4)$ must split into two linear forms, as claimed.

For the case of type II, in the above argument for the case of type III, we replace the role of the fourth double conic \mathcal{C}_4 by the third one \mathcal{C}_3 , and also the role of the remaining double conics \mathcal{C}_i ($i = 1, 2, 3$) by \mathcal{C}_1 and \mathcal{C}_2 . This gives a similar expression

$$(5.10) \quad F - Q^2 = z_3 f_1 f_2 f + g,$$

where f_i ($i = 1, 2$) is a defining equation of the line connecting the points λ_3 and λ_i , f is a linear polynomial in z_0, \dots, z_4 , and $g \in I_Y$. Further, from (5.6) we have $F - Q^2 \in (z_4) + I_Y$. Hence we can write

$$z_3 f_1 f_2 f = z_4 h + g_1, \quad g_1 \in I_Y.$$

If we write $f = cz_4 + \zeta$ where $c \in \mathbb{C}$ and ζ is a linear polynomial without the variable z_4 , we obtain $z_4(cz_3 f_1 f_2 - h) = -z_3 f_1 f_2 \zeta + g_1$. Therefore, since $f_1 f_2 \zeta$ does not involve z_4 , we have $\zeta = 0$, and we obtain $f = cz_4$. By (5.10) this means

$$F - Q^2 = cz_3 z_4 f_1 f_2 + g, \quad g \in I_Y.$$

If $c = 0$, then $F - Q^2 \in I_Y$, which means $F|_Y = Q^2|_Y$. This means that the branch divisor of the double covering $\Phi_3 : Z_3 \rightarrow Y$ is non-reduced. But of course this cannot happen since the restriction of Φ_3 to a general fiber of $\pi : Y \rightarrow \Lambda$ is a non-reduced quartic curve (Proposition 2.3). Therefore we have $c \neq 0$, and we obtain $F - Q^2 = z_3 z_4 f_1 f_2 + g$ with $g \in I_Y$. Then by the same argument in the case of type III, we can normalize the homogeneous coordinates (z_0, z_1, z_2) in a way that $f_1 = z_0$ and $f_2 = z_1$. This means that a defining equation of the hyperquartic \mathcal{B} can be taken in the form (5.1) in Theorem 5.3. The remaining claim about Q follows in exactly the same way as in the case of type III, if we use Lemma 4.3 (iii) instead of Proposition 4.7 (ii). \square

6. DIMENSION OF THE MODULI SPACES

In this section we compute dimension of the moduli space of the present twistor spaces. For the case of type I, this was done in [4, Section 5.1], but the argument in the paper does not work in the cases of type II and type III.

Let Z be a twistor space on $4\mathbb{CP}^2$ whose anticanonical map is of degree two, and suppose that the type of Z is I, II or III. (We include type I since the present argument reproves a result in [4].) Let $S \in |F|$ be a real irreducible member. Then by a similar argument to [4, Proposition 5.1], for the tangent sheaf of Z we have

$$(6.1) \quad H^i(\Theta_Z) = 0 \text{ for } i \neq 1, \quad h^1(\Theta_Z) = 13.$$

(Note that if Z is of type IV, this is not the case.) Also, it is not difficult to show

$$(6.2) \quad H^i(\Theta_S) = 0 \text{ for } i \neq 1, \quad h^1(\Theta_S) = 10,$$

$$(6.3) \quad h^0(K_S^{-1}) = 1, \quad H^i(K_S^{-1}) = 0 \text{ for } i \neq 0.$$

Let $\Theta_{Z,S}$ denote the subsheaf of Θ_Z consisting of germs of vector fields which are tangent to S , and write $\Theta_Z(-S) := \Theta_Z \otimes \mathcal{O}_Z(-S)$. Then by various standard exact sequences of sheaves including these, and noting $N_{S/Z} \simeq K_S^{-1}$, we obtain the following commutative diagram of cohomology groups on Z and S :

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & H^0(\Theta_Z|_S) & \longrightarrow & H^0(K_S^{-1}) & \longrightarrow & H^1(\Theta_S) \longrightarrow H^1(\Theta_Z|_S) \\
 & & \downarrow & & \downarrow \delta & & \parallel \\
 0 & \longrightarrow & H^1(\Theta_Z(-S)) & \longrightarrow & H^1(\Theta_{Z,S}) & \xrightarrow{\alpha} & H^1(\Theta_S) \longrightarrow 0 \\
 & & \downarrow & & \downarrow \beta & & \downarrow \\
 0 & \longrightarrow & H^1(\Theta_Z) & \xlongequal{\quad} & H^1(\Theta_Z) & \longrightarrow & 0 \\
 & & & & \downarrow & & \\
 & & & & 0 & &
 \end{array}$$

From the middle column of this diagram we obtain $h^1(\Theta_{Z,S}) = 14$ by (6.1) and (6.3), which means $h^1(\Theta_Z(-S)) = 4$ from the middle row and (6.2). In particular, the Kuranishi family of deformations of the pair (Z, S) is 14-dimensional.

In order to compute the dimension of the moduli spaces, we first recall that our twistor spaces (of types I–III) can be characterized by the property that they have a rational surface S with particular structure as a member of the system $|F|$. For type II twistor spaces the structure of S is described in the proof of Proposition 3.1 (i) in terms of blowup points of the birational morphism $\epsilon : S \rightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$. In particular, all freedom for S that can be contained in the twistor spaces of type II is to move four points on a fixed cycle of anticanonical curve on $\mathbb{CP}^1 \times \mathbb{CP}^1$ (the remaining four points does not contribute for deforming complex structure), and therefore they constitute 4-dimensional family of deformations of S . Via the Kodaira-Spencer map, this family determines a 4-dimensional subspace of $H^1(\Theta_S)$, for which we denote by V .

We have $\dim \alpha^{-1}(V) = \dim V + h^1(\Theta_Z(-S)) = 8$. The tangent space of the moduli space of twistor spaces of type II can be considered as the space $\beta(\alpha^{-1}(V)) \subset H^1(\Theta_Z)$. The image of the map δ in the diagram corresponds to deformations of (Z, S) that can be obtained by moving S in Z , and of course they do not give a non-trivial deformation of Z . But from the characterization of Z by the complex structure of S , even if we move S in Z , its complex structure cannot go away from the above 4-dimensional family of S . This means that the image of δ is contained in $\alpha^{-1}(V)$. Thus the tangent space of the moduli space of twistor spaces of type II can be identified with the quotient space

$$(6.4) \quad \alpha^{-1}(V)/\delta H^0(K_S^{-1}),$$

and this is 7-dimensional by (6.3).

For the case of type III, the same argument works except that the subspace $V \subset H^1(\Theta_S)$ becomes 2-dimensional in this case. Consequently the tangent space of the moduli space is again identified with the quotient space (6.4), which is 5-dimensional this time.

This way we conclude that the moduli space of twistor spaces is 7-dimensional for the case of type II, and 5-dimensional for the case of type III. We note that as obtained in [4,

Section 5.1], for the case of type I the moduli space is 9-dimensional. This can also be seen from the above argument if we notice $\dim V = 6$ for the case of type I. Thus according to the degenerations $I \rightarrow II$ and $II \rightarrow III$, the dimension of the moduli spaces drops by two. On the other hand for the case of type IV the moduli space is 4-dimensional as obtained in [3]. This discrepancy comes from the fact that twistor spaces of type IV admit a non-trivial \mathbb{C}^* -action, while other three types of spaces do not.

Remark 6.1. Looking the equations of the quartic hypersurfaces in Theorem 5.3, one may think that the dimension of the moduli spaces computed above contradicts Theorem 5.3, because the number of parameters involved in the equation (5.2) in the case of type III is greater than those for the equation (5.1) in the case of type II. But it is not correct, since the elements in $\mathrm{PGL}(5, \mathbb{C})$ preserving the form of (5.2) is greater than those for (5.1).

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